Computer Graphics Worksheet
Sampling and Reconstruction

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Problem 1. Convolution vs Multiplication

The convolution of a function $f(x)$ with a second function $g(x)$ is defined as:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t) \cdot g(x - t)dt$$

The multiplication of two function is defined as the point-wise multiplication:

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

The transformation of a signal $f(x)$ to Fourier space is given by:

$$F(\omega) = \int_{-\infty}^{\infty} f(x) \cdot e^{-2\pi i \omega x} dx$$

We call $\mathscr{F}$ the operator mapping f to Fourier space: $\mathscr{F}[f(x)] = F(\omega)$. Show that convolving in signal space is the same as multiplication in Fourier space:

$$\mathscr{F}[(f * g)(x)] = \mathscr{F}[f(x)] \cdot \mathscr{F}[g(x)]$$

Problem 2. Fourier Transformation

Show that the Fourier transformation of the box function $B_d(x)$ is a *sinc* type function. The sinc function is defined as $sinc(x) = \dfrac{sin \pi x}{\pi x}$ and a definition of the Fourier transform can be found in the Problem 1.

$$B_d(x) = \begin{cases} 0 & \text{for} \quad x \leq -d \\ 1 & \text{for} \quad -d < x < d \\ 0 & \text{for} \quad d \leq x \end{cases}$$

Problem 3. Triangle Filter

Show that reconstructing a signal that is sampled at sampling distance 1 with the triangle filter $T(x)$ is equivalent of performing linear interpolation.

$$T_1(x) = \begin{cases} 0 & \text{for} \quad x < -1 \\ x+1 & \text{for} \quad -1 \le x < 0 \\ -x+1 & \text{for} \quad 0 \le x < 1 \\ 0 & \text{for} \quad 1 \le x \end{cases}$$

Problem 4. Sampling Theory

Let $f(x)$ be an infinite signal that fulfils the Nyqvist property, thus the highest frequency of the signal is smaller than $\frac{1}{2\Delta x}$ if Δx is the sampling distance. Consider a regular sampling $f_s(x)$ of $f(x)$ with sample distance Δx.

a) Is an exact signal reconstruction of $f(x)$ possible? If so, why?
b) How has the reconstruction to be performed in image and Fourier space?

Problem 5. Antialiasing

c) Describe what aliasing means in Fourier space.
d) Consider an infinite signal $f(x)$ and a regular sampling $f_s(x)$ with sampling distance Δx that shows no aliasing artefacts. The sampling distance is now increased step by step until the first aliasing artefacts occur.
   How can we best get *an aliasing-free* sampled signal from these samples? Describe the filter procedure in Fourier and signal space. You do not have to derive the exact filter kernels (but you can of course).